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On Weierstrass compact pseudometric spaces and a weak form of the axiom of choice

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Abstract

We show that the countable multiple choice axiom CMC is equivalent to the assertion: Weierstrass compact pseudometric spaces are compact. © 2000 Elsevier Science B.V. All rights reserved.

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The *countable axiom of choice* CAC (Form 8 in [3]) is the assertion:

For every set $\mathcal{A} = \{A_i \colon i \in \omega\}$ *of nonempty disjoint sets there exists a set* C *consisting of one and only one element from each element of* $\mathcal{A}$.

The *countable multiple axiom of choice*, CMC, is the proposition:

Every set $\mathcal{A} = \{A_i \colon i \in \omega\}$ *of nonempty disjoint sets has a countable multiple choice, i.e., a family* $\mathcal{F} = \{F_i \colon i \in \omega\}$ *of finite nonempty sets such that for every* $i \in \omega$, $F_i \subseteq A_i$.

Let (X, T) be a topological space. X is said to be *countably compact* iff every countable open cover $\mathcal{U}$ of X has a finite subcover $\mathcal{V}$. X is said to be *Weierstrass compact* iff every infinite set $Y \subset X$ has a limit point a, i.e., every open set O including a meets Y in an infinite set. WCC stands for the statement:

WCC = *Weierstrass-compact pseudometric spaces are compact.*

In [1] it is asked what is the set theoretical status of WCC. The aim of this note is to show that WCC is equivalent to CMC.

Lemma 1. *CMC iff WCCC* (*Weierstrass-compact pseudometric spaces are countably compact*).

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Proof. ($\Rightarrow$) Assume on the contrary and let (X, d) be a Weierstrass-compact but not a countably compact pseudometric space. Fix $\mathcal{U} = \{O_n\colon n \in \omega\}$ an open cover of X without a finite subcover. Clearly,

$$Q = \left\{Q_n = \bigcup\{O_m\colon m \leqslant n\}\colon n \in \omega\right\}$$

is an ascending open cover of X such that no Q_n covers X. Use CMC to pick a sequence of nonempty finite sets $\mathcal{F} = \{F_n \subset X \backslash Q_n\colon n \in \omega\}$. Let $x \in O_n$ be a limit point of $\bigcup \mathcal{F}$. Then O_n must contain infinitely many points of $\bigcup \mathcal{F}$ contradicting the choice of F_n's. Hence $\mathcal{U}$ has a finite subcover and X is countably compact as required.

($\Leftarrow$) Assume on the contrary that WCCC holds but CMC fails.

Claim 1 [2,4]. *CMC iff for every countable family $\mathcal{A}$ of disjoint nonempty sets there exists an infinite set $C \subset \bigcup \mathcal{A}$ such that for every $A \in \mathcal{A}$, $0 \leqslant |C \cap A| < \omega$.*

Proof. ($\Rightarrow$) This is straightforward.

($\Leftarrow$) Fix $\mathcal{A} = \{A_n\colon n \in \omega\}$ a family of disjoint nonempty sets. Put

$$\mathcal{B} = \left\{B_n = \prod_{m \leqslant n} A_m\colon n \in \omega\right\}$$

and let $C = \{c_{n_i}\colon i \in \omega\}$ satisfy the conclusion of CMC for the infinite subfamily $\{B_{n_i}\colon i \in \omega\}$ of $\mathcal{B}$. Based on C and taking projections we can easily construct inductively a set $\mathcal{F} = \{F_n\colon n \in \omega\}$ satisfying CMC for $\mathcal{A}$. $\square$

Fix, by Claim 1, a countable family $\mathcal{A} = \{A_i\colon i \in \omega\}$ of disjoint nonempty sets having no infinite subfamily with a countable multiple choice set. Make $X_i = A_i \cup \{i\}$, $i \notin A_i$ into a pseudometric space by requiring:

$$d_i(x, y) = \begin{cases} 1 & \text{if } [((x = i) \wedge (y \neq x)) \vee ((y = i) \wedge (y \neq x))], \\ 0 & \text{otherwise.} \end{cases}$$

Let X be the product of the family $\{(X_i, T_i)\colon i \in \omega\}$ endowed with the Tychonoff topology T, where T_i is the topology induced by the pseudometric d_i. It can be readily verified that T is the topology induced by the pseudometric d given by

$$d(x, y) = \sum_{n \in \omega} \frac{1}{2^n} d_n\big(x(n), y(n)\big).$$

Claim 2. *X is Weierstrass-compact.*

Proof. Let G be an infinite subset of X. As $\mathcal{A}$ has no partial multiple choice set, it follows that for every $g \in G$ there exists $n_g \in \omega$, $g(i) = i$ for all $i \geqslant n_g$. Put $G_n = \{g \in G\colon g(i) = i$ for all $i \geqslant n\}$. Clearly, for all but finitely many n's G_n is infinite (otherwise $\mathcal{A}$ would have a partial multiple choice set). Fix $n \in \omega$ such that G_n is infinite. Since $X_1 \times X_2 \times \cdots \times X_{n-1}$ is a compact space homeomorphic to the subspace

$$X^n = \left\{x \in X\colon x(i) = i \text{ for all } i \geqslant n\right\}$$

and $G_n \subseteq X^n$, it follows by the compactness of X^n, that G_n has a cluster point $x \in X^n$. It is easy to see that x is a cluster point of G_n in X. $\square$

By Claim 2, X is Weierstrass compact. Hence X is countably compact. Thus $K = \{\pi_i^{-1}[A_i]\colon i \in \omega\}$ being a countable family of closed sets with the finite intersection property, satisfies $\bigcap K \neq \emptyset$ meaning that $\mathcal{A}$ has a choice function which is a contradiction. Hence CMC holds finishing the proof of Lemma 1. $\square$

Lemma 2. *CMC iff every compact pseudometric space (X, d) has a dense subspace Y which is written as a countable union of finite sets.*

Proof. ($\Rightarrow$) Fix (X, d) a compact pseudometric space. Using the compactness of X one can show, as usual, that for every $n > 0$,

$$X_n = \left\{Y \in [X]^{<\omega}\colon d(x, Y) < 1/n \text{ for all } x \in X\right\} \neq \emptyset.$$

Put $Z = \{X_n\colon n \in \omega\}$ and let $F = \{F_n\colon n \in \omega\}$ be a countable multiple choice for Z, i.e., $\emptyset \neq F_n \in [X_n]^{<\omega}$. Clearly $Y = \bigcup\{P_n = \bigcup F_n\colon n \in \omega\}$ is a dense subset of X which is expressed as a countable union of finite sets.

($\Leftarrow$) Fix $\{X_n\colon n \in \omega\}$ a family of disjoint nonempty sets. Put $X = (\bigcup\{X_n\colon n \in \omega\}) \cup \{\infty\}$, $\infty \notin \bigcup\{X_n\colon n \in \omega\}$ and let $d : X \times X \to \mathbb{R}$ be given by $d(x, y) = d(y, x)$ and

$$d(x, y) = \begin{cases} 0 & \text{if } x, y \in X_n \text{ for some } n \in \omega, \\ 1/(n+1) & \text{if } x = \infty \text{ and } y \in X_n \text{ for some } n \in \omega, \\ \max\{1/n, 1/m\} & \text{if } x \in X_m,\ y \in X_n,\ m \neq n. \end{cases}$$

It can be readily verified that (X, d) is a compact pseudometric space (every open set containing ∞ includes all but finitely many X_n's). Let $Y = \bigcup\{Y_m\colon m \in \omega\}$ be the dense set which is guaranteed by the hypothesis. For every $n \in \omega$ let $F_n = Y_{m_n} \cap X_n$ where m_n is the first $v \in \omega$ such that $Y_v \cap X_n \neq \emptyset$. Clearly $\{F_n\colon n \in \omega\}$ is a multiple choice set for the family $\{X_n\colon n \in \omega\}$ finishing the proof of Lemma 2. $\square$

By the same proof of Lemma 2 we can show Corollary 3.

Corollary 3.

(i) *CMC iff every compact pseudometric space (X, d) has a dense subspace Y which is written as a well ordered union of finite sets.*

(ii) *CAC iff every compact pseudometric space (X, d) is separable.*

Theorem 4. *CMC iff WCC.*

Proof. ($\Leftarrow$) Since WCC clearly implies WCCC which in turn is equivalent to CMC, we see that WCC implies CMC.

($\Rightarrow$) It suffices to show that WCCC implies WCC. Fix (X, d) a Weierstrass compact countably compact pseudometric space.

Claim 1. *If X has a dense subset which can be written as a countable union of finite sets then X is compact.*

Proof. Let $Y = \bigcup\{Y_n \in [X]^{<\omega}\colon n \in \omega\}$ be the dense subset of X. For every $n \in \omega$ let $B_n = \{D(y, 1/m)\colon y \in Y_n,\ m > 0\}$. Clearly each B_n is countable and $\mathcal{B} = \bigcup\{B_n\colon n \in \omega\}$ is a base for X. In order to show that X is compact it suffices to show that every open cover $U = \{U_i\colon i \in I\} \subseteq \mathcal{B}$ has a finite subcover. For every $n \in \omega$ let $Q_n = \bigcup G_n$, $G_n = U \cap B_n$. Clearly each G_n is countable and $Q = \{Q_n\colon n \in \omega\}$ is a countable open cover of X. It follows that Q has a finite subcover say, $Q_1, Q_2, \ldots, Q_v$. Since $G = G_1 \cup G_2 \cup \cdots \cup G_v$ is again a countable cover of X it follows that it has a finite subcover $\mathcal{V}$. Clearly $\mathcal{V}$ is a finite subcover of U and X is compact as required. □

Claim 2. *X is totally bounded.*

Proof. Assume the contrary. Then there exists $r > 0$ such that for every $n > 0$, $A_n = \{Y \in [X]^n\colon (\forall x, y \in Y)(x \neq y \to d(x, y) \geqslant r)\} \neq \emptyset$. Put $A = \{A_n\colon n \in \omega\}$ and let $F = \{F_n\colon n \in \omega\}$ be a multiple choice set for A. Put

$$Z = \overline{\bigcup\left\{W_n = \bigcup F_n\colon n \in \omega\right\}}.$$

Arguing as in Claim 1 one can easily see that Z is a compact space. Thus the open cover U by balls of radius $r/2$ has a finite subcover. Assume that $D(x_1, r/2), D(x_2, r/2), \ldots, D(x_v, r/2)$ covers Z. It is easy to see that there exists $j \leqslant v$ and $x, y \in D(x_j, r/2)$ such that $d(x, y) \geqslant r$. This contradiction establishes Claim 2. □

Claim 3. *If X is totally bounded then X is compact.*

Proof. For every $n \in \omega$ put $P_n = \{Y \in [X]^{<\omega}\colon d(x, Y) < 1/n$ for all $x \in X\}$. As X is totally bounded it follows that $P_n \neq \emptyset$. Set $P = \{P_n\colon n \in \omega\}$ and let $F = \{F_n\colon n \in \omega\}$ be a countable multiple choice for P. Clearly $Y = \bigcup(\bigcup F)$ is a dense subset of X that is expressed as a countable union of finite sets. Thus, an application of Claim 1 shows that X is compact finishing the proof of Claim 3 and Theorem 4. □

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